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On the role of symmetry in Saint-Venant's problem

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Abstract

In the relaxed Saint-Venant's elastic problem, in virtue of Saint-Venant's Postulate, the pointwise assignments of tractions at cylinder plane ends are replaced by the assignments of the corresponding resultant forces and moments. The solution indeterminacy so introduced is traditionally overcome by postulating that some specific features characterize the elastic state. In this work a relaxed incremental equilibrium problem is posed for a heterogenous anisotropic cylinder, whose tangent elasticity tensor field possesses the usual major and minor symmetries, is positive definite, independent from the axial coordinate and endowed with a plane of elastic symmetry orthogonal to the cylinder axis. Symmetry has been consistently employed to formulate the basic problems of extension, bending, torsion and flexure as symmetric and antisymmetric problems respectively. It is shown that Saint-Venant's postulate, momentum balance and symmetry are sufficient, without resorting to any a priori assumption, to determine the general form of the displacement field and to remove the solution indeterminacy.

Keywords: Saint-Venant; Anisotropic beam

1. Introduction

The elastic problem commonly associated with Saint-Venant concerns the equilibrium of a homogeneous isotropic linear elastic cylinder of finite length subject only to prescribed end loads. de Saint-Venant (1855, 1856) conjectured that except in small neighborhoods of the plane ends, the elastic state depends only upon the resultant end forces and moments, and this enabled him to derive an approximate solution to the problem. The conjecture, which here is termed Saint-Venant's postulate, appears to be applicable in contexts more general than that originally envisaged, and the principle has been confirmed by several authors among the earliest of whom are Toupin (1964) and Fichera (1978), who both established exponential decay estimates.

In virtue of Saint-Venant's postulate, the pointwise assignment of tractions at the plane ends is replaced by the assignment of the corresponding resultant forces and moments, leading to the four basic problems of

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extension, bending, torsion and flexure. The solution of this relaxed equilibrium problem is classically gained by means of the so called semi-inverse method: first, it is postulated that, outside the above small neighborhoods of the plane ends, certain features characterize the elastic state and next the remaining features are determined, such as to satisfy the equations of elasticity. Once a characterization has been postulated and the solution determined, in virtue of Saint-Venant's postulate one is allowed to state that this solution is unique (neglecting border effects nearby the plane ends).

Different general characterizations of Saint-Venant's solution were established by various authors. Clebsh (1862) proved that Saint-Venant's solution can be determined by assuming that no normal stresses are exerted on plane elements parallel to the cylinder longitudinal axis. Voigt (1887) adopted a different assumption regarding the stress field: Saint-Venant's extension, bending and torsion solutions can be obtained by admitting that stress field does not depend on the axial coordinate, while the flexure solution can be determined by assuming that the stress field depends at most linearly on the axial coordinate. Other characterizations of Saint-Venant's solution are related to specific minimum strain energy properties (Sternberg and Knowles, 1966).

By adopting the semi-inverse method, solutions of this relaxed equilibrium problem for particular heterogenous and anisotropic elastic materials were given in various papers (i.e. Lekhnitskii, 1963, 1971; Lomakin, 1976).

A rational scheme for determining Saint-Venant's solution without resorting to any a priori assumption was provided by Iesan (1986, 1987), who considered heterogenous anisotropic materials with elastic properties independent from the axial coordinate. It was proved that, neglecting border effects at the plane ends, the derivative with respect to the axial coordinate of a displacement field satisfying the Saint-Venant's relaxed problem is a rigid displacement field in the cases of extension, bending and torsion, while it is a solution of a bending problem in the case of flexure. The general form of displacement field was derived in the particular case of homogenous elastic material.

In the present work a new method of construction of Saint-Venant's solution, without resorting to any a priori assumption, is provided. The role of symmetry is consistently emphasized, and a relaxed Saint-Venant's equilibrium problem for a heterogeneous linearized anisotropic elastic prismatic cylinder of finite length is treated. An incremental equilibrium state, superimposed to an initial equilibrium state, is considered. It is supposed that the material possesses a tangent (linearized) fourth tangent elasticity tensor field endowed with the usual major and minor symmetries, but additionally it is supposed positive-definite, independent of at least one coordinate variable and with a plane of elastic symmetry orthogonal to the coordinate direction corresponding to this variable. The cylinder is chosen with its axis parallel to the same direction. First, a result similar to that proved by Iesan is alternatively established by appealing to Saint-Venant's postulate and also to momentum balance laws. Next, this result, combined with the assumed symmetries, is employed to derive the general form of the displacement field, which in turn leads by standard methods to a complete solution. Finally, a condition is derived that implies the assumption adopted by Clebsch for the stress field.

2. Relaxed incremental equilibrium problem

Select a cartesian system of orthogonal coordinates $Oxyz$ with unit coordinate vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and let us consider a second-order reflection tensor $\mathbf{S} = \mathbf{I} - 2\mathbf{k} \otimes \mathbf{k}$, in which $\mathbf{I}$ is the second-order identity tensor and $(\mathbf{k} \otimes \mathbf{k})\mathbf{x} = (\mathbf{k} \cdot \mathbf{x})\mathbf{k}$. The reflection $\mathbf{S}$ carries a point $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ into its reflection in the Oxy–plane

$$\mathbf{S}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k} \tag{1}$$

and has the following properties:

$$\mathbf{S}^{\mathrm{T}} = \mathbf{S}, \quad \mathbf{S}^2 = \mathbf{I} \tag{2}$$

We next suppose that a heterogeneous anisotropic elastic material possesses a tangent (linearized) fourth-order elastic tensor $\mathbb{C}_{(\mathbf{x})}$ and let us denote by $\dot{\mathbf{E}}$ the second-order incremental strain tensor. We assume that the tangent elastic tensor field has the following properties:

I. The reflection $\mathbf{S}$ is a symmetry transformation of the material

$$\forall \dot{\mathbf{E}} \in \mathrm{Sym} \quad \mathbf{S}\mathbb{C}_{(\mathbf{x})}[\dot{\mathbf{E}}]\mathbf{S} = \mathbb{C}_{(\mathbf{x})}[\mathbf{S}\dot{\mathbf{E}}\mathbf{S}].$$

II. The tangent elasticity tensor field $\mathbb{C}_{(\mathbf{x})} \in C^2(\Omega)$ does not depend on the axial coordinate

$$\forall \dot{\mathbf{E}} \in \mathrm{Sym} \quad \mathbb{C}_{(\mathbf{x})}[\dot{\mathbf{E}}] = \mathbb{C}_{(x,y)}[\dot{\mathbf{E}}]$$

III. $\mathbf{C}_{(\mathbf{x})}$ is symmetric and positive definite[1]

$$\forall \dot{\mathbf{E}}^{\mathrm{I}} \in \mathrm{Sym} \setminus \{\mathbf{0}\} \quad \dot{\mathbf{E}}^{\mathrm{I}} \cdot \mathbb{C}_{(\mathbf{x})}[\dot{\mathbf{E}}^{\mathrm{I}}] > 0,$$
$$\forall \dot{\mathbf{E}}^{\mathrm{II}} \in \mathrm{Sym} \setminus \{\mathbf{0}\} \quad \dot{\mathbf{E}}^{\mathrm{I}} \cdot \mathbb{C}_{(\mathbf{x})}[\dot{\mathbf{E}}^{\mathrm{II}}] = \dot{\mathbf{E}}^{\mathrm{II}} \cdot \mathbb{C}_{(\mathbf{x})}[\dot{\mathbf{E}}^{\mathrm{I}}].$$

Hypothesis I implies the following restrictions on the Cartesian components of $\mathbb{C}_{(\mathbf{x})}$. In particular, we have

$$\begin{aligned} &\mathbb{C}_{(\mathbf{x})\alpha\beta3\gamma} = \mathbb{C}_{(\mathbf{x})3\gamma\alpha\beta} = \mathbb{C}_{(\mathbf{x})\gamma3\alpha\beta} = 0, \\ &\mathbb{C}_{(\mathbf{x})333\gamma} = \mathbb{C}_{(\mathbf{x})3\gamma33} = \mathbb{C}_{(\mathbf{x})\gamma333} = 0, \end{aligned} \tag{3}$$

where $\alpha, \beta, \gamma = 1, 2$ and the indexes 1, 2, 3 are associated, respectively, to the coordinates x, y, z.

We now let the elastic material occupy a prismatic cylinder $\Omega_{(-l,l)} = \Sigma \times [-l, l]$, of finite length $2l$, with a simply connected smooth regular plane cross-section Σ. The cylinder axis lies along the Oz coordinate direction, the central cross-section lies in the plane Oxy and it is arbitrary placed on it. The plane ends of the cylinder, with axial coordinates $z = l$ and $z = -l$, are respectively denoted by Σ_l and Σ_{-l}; the lateral surface is denoted as $\partial\Omega_l$; the interior of $\Omega_{(-l,l)}$ is denoted by $\Omega^0_{(-l,l)}$. The symmetry of $\Omega_{(-l,l)}$ with respect to a reflection about the middle cross-section enables us to conclude that

$$\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \in \Omega_{(-l,l)} \iff \mathbf{S}\mathbf{x} = x\mathbf{i} + y\mathbf{j} - z\mathbf{k} \in \Omega_{(-l,l)}. \tag{4}$$

On both plane ends of the cylinder, a system of *unspecified* external incremental forces $\dot{\mathbf{p}}^+$, $\dot{\mathbf{p}}^-$ are pointwise distributed, that respectively correspond to the prescribed resultant force and moment $\dot{\mathbf{R}}^+(l)$, $\dot{\mathbf{M}}^+(l)$ on Σ_l and $\dot{\mathbf{R}}^-(-l)$, $\dot{\mathbf{M}}^-(-l)$ on Σ_{-l}, which, of course, are in overall equilibrium. The following relaxed incremental problem is then posed:

$$\dot{\mathbf{E}} = \frac{1}{2}(\nabla\dot{\mathbf{u}} + \nabla^{\mathrm{T}}\dot{\mathbf{u}}) \quad \text{in } \Omega_{(-l,l)}, \tag{5a}$$

$$\dot{\mathbf{T}} = \mathbb{C}_{(\mathbf{x})}[\dot{\mathbf{E}}] \quad \text{in } \Omega_{(-l,l)}, \tag{5b}$$

$$\operatorname{div}\dot{\mathbf{T}} = \mathbf{0} \quad \text{in } \Omega^0_{(-l,l)}, \quad \dot{\mathbf{T}}(n_x\mathbf{i} + n_y\mathbf{j}) = \mathbf{0} \quad \text{in } \partial\Omega_l, \tag{5c}$$

$$\begin{aligned} &\dot{\mathbf{R}}^+(l) = \int_{\Sigma_l} \dot{\mathbf{T}}\boldsymbol{k}\,\mathrm{d}A, \quad \dot{\mathbf{M}}^+(l) = \int_{\Sigma_l} (x\mathbf{i} + y\mathbf{j}) \wedge \dot{\mathbf{T}}\boldsymbol{k}\,\mathrm{d}A, \\ &\dot{\mathbf{R}}^-(-l) = -\int_{\Sigma_{-l}} \dot{\mathbf{T}}\boldsymbol{k}\,\mathrm{d}A, \quad \dot{\mathbf{M}}^-(-l) = -\int_{\Sigma_{-l}} (x\mathbf{i} + y\mathbf{j}) \wedge \dot{\mathbf{T}}\boldsymbol{k}\,\mathrm{d}A, \end{aligned} \tag{5d}$$

$$\dot{\mathbf{u}}(\mathbf{0}) = \mathbf{0}, \quad \nabla\dot{\mathbf{u}}(\mathbf{0}) - \nabla^{\mathrm{T}}\dot{\mathbf{u}}(\mathbf{0}) = \mathbf{0} \tag{5e}$$

where any kind of geometrical non linearity is neglected; $\dot{\mathbf{T}}$ is the incremental Cauchy stress tensor; $\dot{\mathbf{u}}$ the incremental displacement vector.

The kinematic constraints (5e), although somewhat arbitrarily chosen, exclude rigid body displacements and are invariant with respect to the reflection $\mathbf{S}$. We assume that a solution

$$\dot{\mathbf{u}} \in C^2(\Omega^0) \cap C^1(\Omega) \tag{6}$$

of the above problem exists. It should be noted that the above assumption is very strong and that it is really difficult to prove regularity up to boundary for solutions of the boundary value problem (5a)–(5e) in a Lipschitz domain (Dahlberg and Kening, 2004). However, assumption (6) is common in the literature on the Saint-Venant's problem, and therefore it has been adopted here.

[1] The inner product of two second-order tensor is defined as $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}(\mathbf{A}^{\mathrm{T}}\mathbf{B})$.

3. A property of the solutions of the relaxed Saint-Venant's equilibrium problem

Let us note that contributions to the study of Saint-Venant's postulate, surveyed notably in the review articles by Horgan and Knowles (1981), and by Horgan (1989, 1996), mostly utilize assumption III, which is therefore essential for the later developments.

In particular, we now use Saint-Venant's postulate, together with assumption II and momentum balance laws to determine the dependence of the incremental strain and stress fields on the axial coordinate z.

With this aim, we consider the cylinder $\Omega_{(-l,l)}$ as part of a larger cylinder $\Omega_{(-L,L)} = \Sigma \times [-L, L]$, with $L > 2l$ (Fig. 1), whose lateral surface is traction-free. On the plane ends of $\Omega_{(-L,L)}$ unspecified incremental tractions are applied, statically equivalent to the prescribed incremental resultant forces $\dot{\mathbf{R}}^{+}(L)$, $\dot{\mathbf{R}}^{-}(-L)$ and moments $\dot{\mathbf{M}}^{+}(L)$, $\dot{\mathbf{M}}^{-}(-L)$. The external forces are in overall equilibrium and the following relaxed base equilibrium conditions hold:

$$\begin{aligned}
&\dot{\mathbf{R}}^{+}(L) = \int_{\Sigma_L} \dot{\mathbf{T}}\boldsymbol{k}\,\mathrm{d}A, \quad \dot{\mathbf{M}}^{+}(L) = \int_{\Sigma_L} (x\mathbf{i} + y\mathbf{j}) \wedge \dot{\mathbf{T}}\boldsymbol{k}\,\mathrm{d}A, \\
&\dot{\mathbf{R}}^{-}(-L) = -\int_{\Sigma_{-L}} \dot{\mathbf{T}}\boldsymbol{k}\,\mathrm{d}A, \quad \dot{\mathbf{M}}^{-}(-L) = -\int_{\Sigma_{-L}} (x\mathbf{i} + y\mathbf{j}) \wedge \dot{\mathbf{T}}\boldsymbol{k}\,\mathrm{d}A.
\end{aligned} \tag{7}$$

By virtue of momentum balance and lateral boundary conditions, in a generic internal section Σ_z the internal resultant force $\dot{\mathbf{R}}^{+}(z)$ and resultant moment $\dot{\mathbf{M}}^{+}(z)$ must satisfy the equilibrium relations

$$\begin{aligned}
&\dot{\mathbf{R}}^{+}(z) = \dot{\mathbf{R}}^{+}(0), \\
&\dot{\mathbf{M}}^{+}(z) = \dot{\mathbf{M}}^{+}(0) + z\dot{\mathbf{R}}^{+} \wedge \mathbf{k},
\end{aligned} \tag{8}$$

which imply that the internal resultant force $\dot{\mathbf{R}}^{+}$ is constant in z. Now, let

$$\dot{\mathbf{E}}(x, y, \Delta z)_{\{-l,\, l,\, \dot{\mathbf{R}}^{+}, \dot{\mathbf{M}}^{+}(-l), \dot{\mathbf{M}}^{+}(l)\}} \tag{9}$$

be the incremental strain tensor in a material point $(x, y, \Delta z) \in \Omega_{(-l,l)}$ that, by virtue of Saint-Venant's postulate outside some small neighboring of plane ends Σ_l and Σ_{-l} is completely determined, with negligible error, by the internal resultant force $\dot{\mathbf{R}}^{+}$ and by the internal resultant moments $\dot{\mathbf{M}}^{+}(-l)$ and $\dot{\mathbf{M}}^{+}(l)$.

Next, a cylinder $\Omega_{(-l+\Delta z, l+\Delta z)} \subset \Omega_{(-L,L)}$, with $|\Delta z| < l$, is considered. Since $(x, y, \Delta z) \in \Omega_{(-l,l)} \cap \Omega_{(-l+\Delta z, l+\Delta z)}$, it follows that:

$$\dot{\mathbf{E}}(x, y, \Delta z)_{\{-l,\, l,\, \dot{\mathbf{R}}^{+}, \dot{\mathbf{M}}^{+}(-l), \dot{\mathbf{M}}^{+}(l)\}} = \dot{\mathbf{E}}(x, y, \Delta z)_{\{-l+\Delta z,\, l+\Delta z,\, \dot{\mathbf{R}}^{+}, \dot{\mathbf{M}}^{+}(-l)+\Delta z\dot{\mathbf{R}}^{+}\wedge\mathbf{k}, \dot{\mathbf{M}}^{+}(l)+\Delta z\dot{\mathbf{R}}^{+}\wedge\mathbf{k}\}}, \tag{10}$$

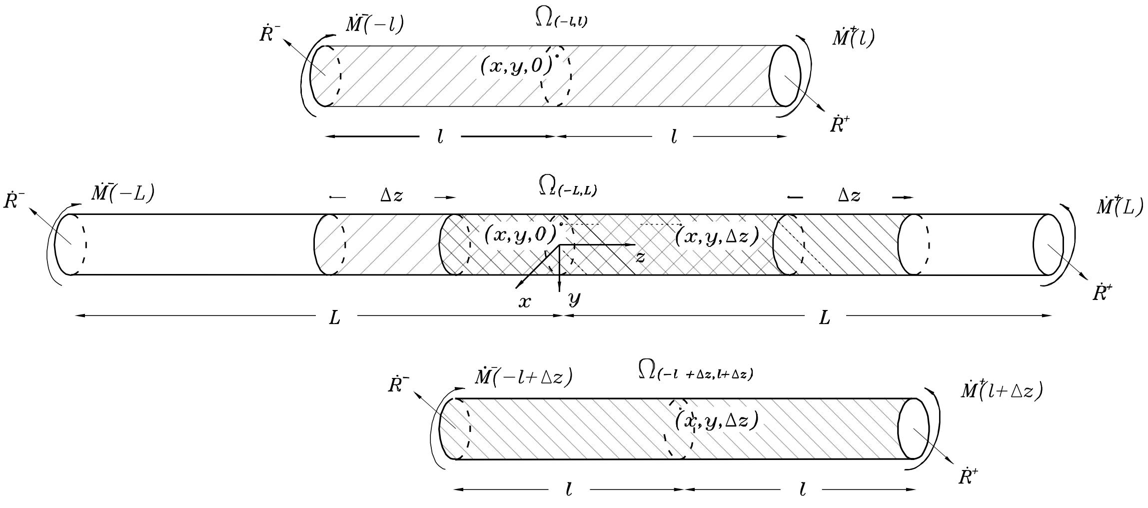

Fig. 1. The cylinders $\Omega_{(-l,l)}$, $\Omega_{(-L,L)}$, $\Omega_{(-l+\Delta z, l+\Delta z)}$.

where from the superposition principle, we get

$$\dot{\mathbf{E}}(x,y,\Delta z)_{\{-l+\Delta z,\, l+\Delta z,\, \dot{\mathbf{R}}^+,\, \dot{\mathbf{M}}^+(-l)+\Delta z\dot{\mathbf{R}}^+\wedge\mathbf{k},\, \dot{\mathbf{M}}^+(l)+\Delta z\dot{\mathbf{R}}^+\wedge\mathbf{k}\}} = \dot{\mathbf{E}}(x,y,\Delta z)_{\{-l+\Delta z,\, l+\Delta z,\, \dot{\mathbf{R}}^+,\, \dot{\mathbf{M}}^+(-l),\, \dot{\mathbf{M}}^+(l)\}} + \Delta z\dot{\mathbf{E}}(x,y,\Delta z)_{\{-l+\Delta z,\, l+\Delta z,\, \mathbf{0},\, \dot{\mathbf{R}}^+\wedge\mathbf{k},\, \dot{\mathbf{R}}^+\wedge\mathbf{k}\}}. \tag{11}$$

Further, in view of Saint-Venant's postulate and assumption II we can state that, with negligible error, the following equalities hold:

$$\dot{\mathbf{E}}(x,y,\Delta z)_{\{-l+\Delta z,\, l+\Delta z,\, \dot{\mathbf{R}}^+,\, \dot{\mathbf{M}}^+(-l),\, \dot{\mathbf{M}}^+(l)\}} = \dot{\mathbf{E}}(x,y,0)_{\{-l,\, l,\, \dot{\mathbf{R}}^+,\, \dot{\mathbf{M}}^+(-l),\, \dot{\mathbf{M}}^+(l)\}}, \tag{12}$$

$$\dot{\mathbf{E}}(x,y,\Delta z)_{\{-l+\Delta z,\, l+\Delta z,\, \mathbf{0},\, \dot{\mathbf{R}}^+\wedge\mathbf{k},\, \dot{\mathbf{R}}^+\wedge\mathbf{k}\}} = \dot{\mathbf{E}}(x,y,0)_{\{-l,\, l,\, \mathbf{0},\, \dot{\mathbf{R}}^+\wedge\mathbf{k},\, \dot{\mathbf{R}}^+\wedge\mathbf{k}\}}. \tag{13}$$

Hence we conclude that

$$\dot{\mathbf{E}}(x,y,\Delta z)_{\{-l,\, l,\, \dot{\mathbf{R}}^+,\, \dot{\mathbf{M}}^+(-l),\, \dot{\mathbf{M}}^+(l)\}} = \dot{\mathbf{E}}(x,y,0)_{\{-l,\, l,\, \dot{\mathbf{R}}^+,\, \dot{\mathbf{M}}^+(-l),\, \dot{\mathbf{M}}^+(l)\}} + \Delta z\dot{\mathbf{E}}(x,y,0)_{\{-l,\, l,\, \mathbf{0},\, \dot{\mathbf{R}}^+\wedge\mathbf{k},\, \dot{\mathbf{R}}^+\wedge\mathbf{k}\}}, \tag{14}$$

so establishing that the derivative of the incremental strain field with respect to the axial coordinate Δz is equal to the incremental strain field induced by the bending moment $\dot{\mathbf{R}}^+ \wedge \mathbf{k}$.

In the particular case of axial resultant force $\dot{\mathbf{R}}^+ = \dot{N}\mathbf{k}$ applied, the internal resultant moment is constant and (14) becomes

$$\dot{\mathbf{E}}(x,y,\Delta z)_{\{-l,\, l,\, \dot{N}\mathbf{k},\, \dot{\mathbf{M}}^+,\, \dot{\mathbf{M}}^+\}} = \dot{\mathbf{E}}(x,y,0)_{\{-l,\, l,\, \dot{N}\mathbf{k},\, \dot{\mathbf{M}}^+,\, \dot{\mathbf{M}}^+\}}, \tag{15}$$

which shows that in this particular case the incremental strain field is independent on the axial coordinate Δz, so exhibiting a *translational symmetry*.

The relations (14) and (15) determine the dependence on the axial coordinate z of the incremental strain field in the cylinder $\Omega_{(-l,l)}$, neglecting the border effects nearby the plane ends Σ_l and Σ_{-l}. These will be applied in the following to determine the general form of the displacement solution of Saint-Venant's problem.

4. Symmetric and antisymmetric relaxed equilibrium problems

The equilibrium problem (5a)–(5e) can be uniquely decomposed into a symmetric problem, in which both the cylinder and the external forces are invariant with respect to a reflection about the middle cross-section, and an antisymmetric one, in which the same reflection produces an inversion of external forces. With this aim, the self-equilibrated *unspecified* incremental surface forces $\dot{\mathbf{p}}^+$ and $\dot{\mathbf{p}}^-$, respectively applied to the plane ends Σ_l and Σ_{-l}, are considered as superposition of the incremental symmetric surface forces

$$\begin{aligned}
\dot{\mathbf{p}}_s^+(x,y) &= \frac{1}{2}(\dot{\mathbf{p}}^+(x,y) + \mathbf{S}\dot{\mathbf{p}}^-(x,y)),\\
\dot{\mathbf{p}}_s^-(x,y) &= \frac{1}{2}(\dot{\mathbf{p}}^-(x,y) + \mathbf{S}\dot{\mathbf{p}}^+(x,y)),
\end{aligned} \tag{16}$$

such that

$$\dot{\mathbf{p}}_s^-(x,y) = \mathbf{S}\dot{\mathbf{p}}_s^+(x,y) \tag{17}$$

and of the incremental *antisymmetric* external forces

$$\dot{\mathbf{p}}_a^+(x,y) = \frac{1}{2}(\dot{\mathbf{p}}^+(x,y) - \mathbf{S}\dot{\mathbf{p}}^-(x,y)), \tag{18}$$

$$\dot{\mathbf{p}}_a^-(x,y) = \frac{1}{2}(\dot{\mathbf{p}}^-(x,y) - \mathbf{S}\dot{\mathbf{p}}^+(x,y)), \tag{19}$$

such that

$$\dot{\mathbf{p}}_a^-(x,y) = -\mathbf{S}\dot{\mathbf{p}}_a^+(x,y). \tag{20}$$

Since the surface forces $\dot{\mathbf{p}}^+$ and $\dot{\mathbf{p}}^-$ are in global equilibrium, the symmetric forces $\dot{\mathbf{p}}_s^+$ and $\dot{\mathbf{p}}_s^-$ and the antisymmetric forces $\dot{\mathbf{p}}_a^+$ and $\dot{\mathbf{p}}_a^-$ also are in global equilibrium. Next, in view of (8) and (17) the *symmetric* surface forces $\dot{\mathbf{p}}_s^+$ and $\dot{\mathbf{p}}_s^-$ are statically equivalent to the following axial external forces and bending moments (Fig. 2):

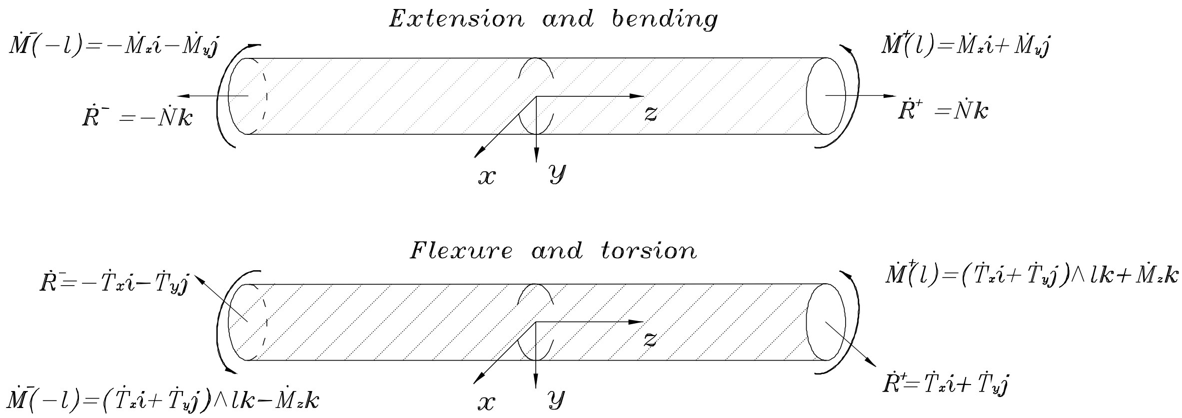

Fig. 2. Symmetric problem and antisymmetric problem.

$$\begin{aligned}
\dot{N}\mathbf{k} &= \int_{\Sigma_l} \dot{\mathbf{p}}_s^+ \,\mathrm{d}A, \quad \dot{M}_x\mathbf{i} + \dot{M}_y\mathbf{j} = \int_{\Sigma_l} (x\mathbf{i} + y\mathbf{j}) \wedge \dot{\mathbf{p}}_s^+ \,\mathrm{d}A, \\
-\dot{N}\mathbf{k} &= \int_{\Sigma_{-l}} \dot{\mathbf{p}}_s^- \,\mathrm{d}A, \quad -\dot{M}_x\mathbf{i} - \dot{M}_y\mathbf{j} = \int_{\Sigma_{-l}} (x\mathbf{i} + y\mathbf{j}) \wedge \dot{\mathbf{p}}_s^- \,\mathrm{d}A.
\end{aligned} \tag{21}$$

Hence, by applying symmetric surface forces, we are led to the basic equilibrium problem of extension and bending.

Meanwhile, in view of (8) and (20) the *antisymmetric* surface forces $\dot{\mathbf{p}}_a^+$ and $\dot{\mathbf{p}}_a^-$ are statically equivalent to the following shearing forces, bending and torsion moments:

$$\begin{aligned}
\dot{T}_x\mathbf{i} + \dot{T}_y\mathbf{j} &= \int_{\Sigma_l} \dot{\mathbf{p}}_a^+ \,\mathrm{d}A, \quad \dot{M}_z\mathbf{k} + (\dot{T}_x\mathbf{i} + \dot{T}_y\mathbf{j}) \wedge l\mathbf{k} = \int_{\Sigma_l} (x\mathbf{i} + y\mathbf{j}) \wedge \dot{\mathbf{p}}_a^+ \,\mathrm{d}A, \\
-\dot{T}_x\mathbf{i} - \dot{T}_y\mathbf{j} &= \int_{\Sigma_{-l}} \dot{\mathbf{p}}_a^- \,\mathrm{d}A, \quad -\dot{M}_z\mathbf{k} + (\dot{T}_x\mathbf{i} + \dot{T}_y\mathbf{j}) \wedge l\mathbf{k} = \int_{\Sigma_{-l}} (x\mathbf{i} + y\mathbf{j}) \wedge \dot{\mathbf{p}}_a^- \,\mathrm{d}A.
\end{aligned} \tag{22}$$

Hence, by applying antisymmetric surface forces, we are led to the basic equilibrium problems of torsion and flexure.

4.1. Extension and bending

The extension and bending solution is here determined by applying the unspecified symmetric incremental surface forces $\dot{\mathbf{p}}_s^+$ and $\dot{\mathbf{p}}_s^- = \mathbf{S}\dot{\mathbf{p}}_s^+$ to the cylinder plane ends Σ_l and Σ_{-l}. Under this condition, if the incremental displacement field $\dot{\mathbf{u}}(x)$ is a solution of the equilibrium problem (5a)–(5e), then the *reflected* field

$$\dot{\mathbf{u}}^*(\mathbf{x}) = \mathbf{S}\dot{\mathbf{u}}(\mathbf{S}\mathbf{x}) \tag{23}$$

is also a solution. In fact, from (2), I and II we get

$$\dot{\mathbf{E}}^*(\mathbf{x}) = (\nabla\dot{\mathbf{u}}^*(x) + \nabla^{\mathrm{T}}\dot{\mathbf{u}}^*(\mathbf{x}))/2 = \mathbf{S}\dot{\mathbf{E}}(\mathbf{S}\mathbf{x})\mathbf{S}, \tag{24}$$

$$\dot{\mathbf{T}}^*(\mathbf{x}) = \mathbb{C}_{(\mathbf{x})}[\mathbf{S}\dot{\mathbf{E}}(\mathbf{S}\mathbf{x})\mathbf{S}] = \mathbf{S}\mathbb{C}_{(\mathbf{S}\mathbf{x})}[\dot{\mathbf{E}}(\mathbf{S}\mathbf{x})]\mathbf{S} = \mathbf{S}\dot{\mathbf{T}}(\mathbf{S}\mathbf{x})\mathbf{S}, \tag{25}$$

$$\operatorname{div}\dot{\mathbf{T}}^*(\mathbf{x}) = \mathbf{S}\operatorname{div}\dot{\mathbf{T}}(\mathbf{S}\mathbf{x}), \tag{26}$$

and, from (4), (2) and (5c)

$$\operatorname{div}\dot{\mathbf{T}}^*(\mathbf{x}) = \mathbf{0} \quad \text{in } \Omega^0_{(-l,l)}, \tag{27}$$

$$\dot{\mathbf{T}}^*(n_x\mathbf{i} + n_y\mathbf{j}) = \mathbf{0} \quad \text{in } \partial\Omega_l, \tag{28}$$

$$\dot{\mathbf{T}}^*\boldsymbol{k} = \dot{p}_s^+ \quad \text{in } \Sigma_l, \quad -\dot{\mathbf{T}}^*\boldsymbol{k} = S\dot{p}_s^+ \quad \text{in } \Sigma_{-l}, \tag{29}$$

$$\dot{\mathbf{u}}^*(\mathbf{0}) = \mathbf{0}, \quad \nabla\dot{\mathbf{u}}^*(\mathbf{0}) - \nabla^{\mathrm{T}}\dot{\mathbf{u}}^*(\mathbf{0}) = \mathbf{0}. \tag{30}$$

Consequently, when $\dot{\mathbf{u}}(\mathbf{x})$ is a solution of (5a)–(5e) then so also is $\dot{\mathbf{u}}^*(\mathbf{x})$. But these solutions each satisfy the same pointwise end conditions (29) and accordingly by uniqueness, for which assumption III is a sufficient condition, these are equal and we have

$$\mathbf{S}\dot{\mathbf{u}}(\mathbf{S}\mathbf{x}) = \dot{\mathbf{u}}(\mathbf{x}), \tag{31}$$

or, in component form

$$\dot{u}(x, y, -z) = \dot{u}(x, y, z), \tag{32}$$

$$\dot{v}(x, y, -z) = \dot{v}(x, y, z), \tag{33}$$

$$\dot{w}(x, y, -z) = -\dot{w}(x, y, z). \tag{34}$$

Condition (31) could seem in some way self-evident, since the cylinder, the tangent elasticity tensor field and the applied forces are invariant under the reflection $\mathbf{S}$, but it should also be noted that a *symmetry breaking* could occur if the cylinder response is not unique. This observation demonstrates the importance of Assumption III, which enables us to conclude (31).

Now, the kinematic conditions (15) and (31) permit the general form of the displacement field to be derived. First, from these we find

$$\mathbf{S}\dot{\mathbf{E}}(x, y)\mathbf{S} = \dot{\mathbf{E}}(x, y), \tag{35}$$

which shows that the antisymmetric incremental strains components $\dot{\epsilon}_{xz}$ and $\dot{\epsilon}_{yz}$, whose sign is changed by reflection $\mathbf{S}$, must vanish. Then, in component form, we can write

$$\begin{bmatrix} \dot{\epsilon}_x(x,y) & \dot{\epsilon}_{xy}(x,y) & 0 \\ \dot{\epsilon}_{xy}(x,y) & \dot{\epsilon}_y(x,y) & 0 \\ 0 & 0 & \dot{\epsilon}_z(x,y) \end{bmatrix} = \begin{bmatrix} \dot{u}_{,x} & \frac{(\dot{u}_{,y}+\dot{v}_{,x})}{2} & \frac{(\dot{w}_{,x}+\dot{v}_{,z})}{2} \\ \frac{(\dot{u}_{,y}+\dot{v}_{,x})}{2} & \dot{v}_{,y} & \frac{(\dot{w}_{,y}+\dot{v}_{,z})}{2} \\ \frac{(\dot{w}_{,x}+\dot{v}_{,z})}{2} & \frac{(\dot{w}_{,y}+\dot{v}_{,z})}{2} & \dot{w}_{,z} \end{bmatrix}. \tag{36}$$

Next, (6) and (34) imply that

$$\dot{w}(x, y, 0^+) = -\dot{w}(x, y, 0^-) = 0 \tag{37}$$

showing that the middle cross-section remains plane during deformation. Then, from (36) and (37) we find

$$\dot{w}(x, y, z) = \dot{\epsilon}_z(x, y)z, \tag{38}$$

moreover, from (38) and (36)

$$2\dot{\epsilon}_{xz}(x, y) = \dot{u}_{,z}(x, y, z) + \dot{\epsilon}_{z,x}(x, y)z = 0, \tag{39}$$

from which, by integration with respect to z, we obtain

$$\dot{u}(x, y, z) = \dot{u}(x, y, 0) - \dot{\epsilon}_{z,x}(x, y)\frac{z^2}{2}, \tag{40}$$

and, similarly, from (38) and (36), we find

$$\dot{v}(x, y, z) = \dot{v}(x, y, 0) - \dot{\epsilon}_{z,y}(x, y)\frac{z^2}{2}. \tag{41}$$

Next, from (36) and (40) it follows that

$$\dot{\epsilon}_x(x, y) = \dot{u}_{,x}(x, y, 0) - \dot{\epsilon}_{z,xx}(x, y)\frac{z^2}{2}, \tag{42}$$

which, by the arbitrariness of z, necessarily implies

$$\dot{\epsilon}_{z,xx}(x,y)=0, \tag{43}$$

similarly, we conclude from (36) and (41) that

$$\dot{\epsilon}_{z,yy}(x,y)=0, \tag{44}$$

while (36), (40) and (41) give

$$2\dot{\epsilon}_{xy}(x,y)=\dot{v}_{,x}(x,y,0)+\dot{u}_{,y}(x,y,0)-\dot{\epsilon}_{z,xy}(x,y)z^2 \tag{45}$$

and consequently

$$\dot{\epsilon}_{z,xy}(x,y)=0. \tag{46}$$

We may now appeal to (43), (44) and (46) to derive the general form of the incremental axial strain field, which becomes

$$\dot{\epsilon}_z(x,y)=\dot{\epsilon}_0+\dot{\kappa}_y x+\dot{\kappa}_x y, \tag{47}$$

where $\dot{\epsilon}_0$ is the unknown incremental deformation of the material points of axis Oy; $\dot{\kappa}_x$ is the unknown incremental curvature along axis Ox, $-\dot{\kappa}_y$ is the unknown incremental curvature along axis Oy. Hence, in view of (40), (41) and (47), the general form of the incremental displacement field is

$$\dot{\mathbf{u}}(\mathbf{x})=\left(\dot{u}(x,y,0)-\dot{\kappa}_y\frac{z^2}{2}\right)\mathbf{i}+\left(\dot{v}(x,y,0)-\dot{\kappa}_x\frac{z^2}{2}\right)\mathbf{j}+(\dot{\epsilon}_0+\dot{\kappa}_y x+\dot{\kappa}_x y)z\mathbf{k}, \tag{48}$$

which depends, besides the unknown parameters $\dot{\epsilon}_0$ and $\dot{\kappa}_y$, $\dot{\kappa}_x$, on the unknown functions $\dot{u}(x,y,0)$, $\dot{v}(x,y,0)$. The general form (48) shows that all cross-sections remain plane and orthogonal to the longitudinal fibers, with bending cross-section rotations $\dot{\kappa}_y z$ and $-\dot{\kappa}_x z$ along the axis Ox and Oy proportional to the axial coordinate z.

Once the general form of the incremental displacement field (48) is known, the indeterminacy of the symmetric relaxed equilibrium problem is overcome and its solution can be determined in a standard way, by solving a generalized plane equilibrium problem. With this aim, we note that I, II and (35) give

$$[\dot{\mathbf{T}}(x,y)]=\begin{bmatrix}\dot{\sigma}_x(x,y) & \dot{\tau}_{xy}(x,y) & 0\\ \dot{\tau}_{xy}(x,y) & \dot{\sigma}_y(x,y) & 0\\ 0 & 0 & \dot{\sigma}_z(x,y)\end{bmatrix}, \tag{49}$$

which shows that antisymmetric stresses must vanish. Next, we adopt the following decomposition of the incremental stress and strain fields:

$$\begin{aligned}\dot{\mathbf{E}}&=\dot{\mathbf{E}}^\circ+\dot{\varepsilon}_z\mathbf{k}\otimes\mathbf{k},\\ \dot{\mathbf{T}}&=\dot{\mathbf{T}}^\circ+\dot{\sigma}_z\mathbf{k}\otimes\mathbf{k},\end{aligned} \tag{50}$$

where $\dot{\mathbf{E}}^\circ$ and $\dot{\mathbf{T}}^\circ$ are plane incremental strain and stress fields. Then, in view of (48), (49), (5a), (5b), (5c) and (5e) we are led to the following plane elasticity problem[2]:

$$\dot{\mathbf{E}}^\circ=\dot{u}_{,x}\mathbf{i}\otimes\mathbf{i}+\dot{v}_{,y}\mathbf{j}\otimes\mathbf{j}+\frac{(\dot{u}_{,y}+\dot{v}_{,x})}{2}(\mathbf{i}\otimes\mathbf{j}+\mathbf{j}\otimes\mathbf{i}), \tag{51a}$$

$$\dot{\mathbf{T}}^\circ=\mathbb{C}^\circ_{(x,y)}[\dot{\mathbf{E}}^\circ]+(\dot{\epsilon}_0+\dot{\kappa}_y x+\dot{\kappa}_x y)[\mathbb{C}_{(x,y)_{1133}}\mathbf{i}\otimes\mathbf{i}+\mathbb{C}_{(x,y)_{2233}}\mathbf{j}\otimes\mathbf{j}+\mathbb{C}_{(x,y)_{1233}}(\mathbf{i}\otimes\mathbf{j}+\mathbf{j}\otimes\mathbf{i})], \tag{51b}$$

$$\dot{\mathbf{T}}^\circ_{,x}\mathbf{i}+\dot{\mathbf{T}}^\circ_{,y}\mathbf{j}=\mathbf{0}\quad\text{in }\Sigma, \tag{51c}$$

$$\dot{\mathbf{T}}^\circ(n_x\mathbf{i}+n_y\mathbf{j})=\mathbf{0}\quad\text{in }\partial\Sigma, \tag{51d}$$

$$\dot{u}(\mathbf{0})=\dot{v}(\mathbf{0})=0,\quad \dot{u}_{,y}(\mathbf{0})=-\dot{v}_{,x}(\mathbf{0}). \tag{51e}$$

[2] $\mathbb{C}^\circ_{(x,y)}$ denotes the plane tangent elasticity tensor with components $\mathbb{C}^\circ_{(x,y)_{\alpha\beta\gamma\delta}}=\mathbb{C}_{(x,y)_{\alpha\beta\gamma\delta}}$ $(\alpha,\beta,\gamma,\delta=1,2)$.

Due to the particular form of (51b), the solution of the above plane problem depends linearly on the unknown kinematic parameters $\dot{\epsilon}_0$, $\dot{\kappa}_x$ and $\dot{\kappa}_y$. Then we can write

$$\dot{u}(x,y,0) = u_\varepsilon(x,y)\dot{\epsilon}_0 + u_{\kappa_x}(x,y)\dot{\kappa}_x + u_{\kappa_y}(x,y)\dot{\kappa}_y, \tag{52}$$

$$\dot{v}(x,y,0) = v_\varepsilon(x,y)\dot{\epsilon}_0 + v_{\kappa_x}(x,y)\dot{\kappa}_x + v_{\kappa_y}(x,y)\dot{\kappa}_y, \tag{53}$$

$$\dot{\mathbf{E}}^\circ(x,y) = \mathbf{E}^\circ_\varepsilon(x,y)\dot{\epsilon}_0 + \mathbf{E}^\circ_{\kappa_x}(x,y)\dot{\kappa}_x + \mathbf{E}^\circ_{\kappa_y}(x,y)\dot{\kappa}_y. \tag{54}$$

where

- $u_\varepsilon(x,y)$, $v_\varepsilon(x,y)$ and $\mathbf{E}^\circ_\varepsilon(x,y)$ are the displacement and strain solutions corresponding to $(\dot{\epsilon}_0, \dot{\kappa}_x, \dot{\kappa}_y) = (1,0,0)$;
- $u_{\kappa_x}(x,y)$, $v_{\kappa_x}(x,y)$ and $\mathbf{E}^0_{\kappa_x}(x,y)$ are the displacement and strain solutions corresponding to $(\dot{\epsilon}_0, \dot{\kappa}_x, \dot{\kappa}_y) = (0,1,0)$;
- $u_{\kappa_x}(x,y)$, $v_{\kappa_x}(x,y)$ and $\mathbf{E}^0_{\kappa_y}(x,y)$ are the displacement and strain solutions corresponding to $(\dot{\epsilon}_0, \dot{\kappa}_x, \dot{\kappa}_y) = (0,0,1)$;
- the null solution corresponds to null values of the parameters.

Finally, the unknown incremental strain $\dot{\epsilon}_0$ and curvatures $\dot{\kappa}_y$, $\dot{\kappa}_x$ are determined by assigning the resultant force and moment in a generic section

$$\dot{N}\mathbf{k} = \int_\Sigma \dot{\sigma}_z(x,y)\mathbf{k}\,\mathrm{d}A, \quad \dot{M}_x\mathbf{i} + \dot{M}_y\mathbf{j} = \int_\Sigma (x\mathbf{i} + y\mathbf{j}) \wedge \dot{\sigma}_z(x,y)\mathbf{k}\,\mathrm{d}A, \tag{55}$$

where in view of (54), $\dot{\sigma}_z(x,y)$ depends linearly on $\dot{\epsilon}_0$, $\dot{\kappa}_x$ and $\dot{\kappa}_y$

$$\dot{\sigma}_z(x,y) = \mathbf{k}\cdot\mathbb{C}_{(x,y)}\Big[\mathbf{E}^\circ_\varepsilon(x,y)\dot{\epsilon}_0 + \mathbf{E}^\circ_{\kappa_x}(x,y)\dot{\kappa}_x + \mathbf{E}^\circ_{\kappa_y}(x,y)\dot{\kappa}_y\Big]\mathbf{k} + \mathbb{C}_{(x,y)3333}(\dot{\epsilon}_0 + \dot{\kappa}_y x + \dot{\kappa}_x y). \tag{56}$$

Now, we search the condition under which the Clebsh characterization (which implies $\dot{\mathbf{T}}^\circ = \mathbf{0}$) holds in the present case. By assuming $\dot{\mathbf{T}}^\circ = \mathbf{0}$, the equilibrium conditions (51c) and (51d) are clearly satisfied and from (51b) we determine the incremental plane strain field

$$\dot{\mathbf{E}}^\circ_*(x,y) = -(\dot{\epsilon}_0 + \dot{\kappa}_y x + \dot{\kappa}_x y)\mathbb{C}^{\circ^{-1}}_{(x,y)}\left[\mathbb{C}_{(x,y)_{1133}}\mathbf{i}\otimes\mathbf{i} + \mathbb{C}_{(x,y)_{2233}}\mathbf{j}\otimes\mathbf{j} + \mathbb{C}_{(x,y)_{1233}}(\mathbf{i}\otimes\mathbf{j} + \mathbf{j}\otimes\mathbf{i})\right]. \tag{57}$$

Then, under the assumption $\dot{\mathbf{T}}^\circ = \mathbf{0}$, a displacement solution of the relaxed equilibrium problem (5a)–(5e) exists if and only if the incremental strain field (57) satisfies the compatibility equation

$$\dot{\epsilon}^\circ_{*x,yy}(x,y) + \dot{\epsilon}^\circ_{*y,xx}(x,y) = 2\dot{\epsilon}^\circ_{*xy,xy}(x,y), \tag{58}$$

which here must be regarded as a condition on the tangent elasticity field $\mathbb{C}_{(x,y)}$. It is easy to note that a particular solution of (58) is given by a tangent elasticity field in the form $\mathbb{C}_{(x,y)} = f(x,y)\overline{\mathbb{C}}$, where $f(x,y)$ is an arbitrary positive smooth function and $\overline{\mathbb{C}}$ is a constant forth-order tensor (which must satisfy assumptions II and III).

4.2. Torsion and flexure

The torsion and flexure solutions are here determined by applying the unspecified antisymmetric incremental surface forces $\dot{\mathbf{p}}^+_a$ and $\dot{\mathbf{p}}^-_a = \mathbf{S}\dot{\mathbf{p}}^+_a$ to the cylinder plane ends Σ_l and Σ_{-l}. In this case, if $\dot{\mathbf{u}}(\mathbf{x})$ is a displacement solution, then so also is $-\dot{\mathbf{u}}^*(\mathbf{x})$. These solutions each satisfy the same *pointwise* end equilibrium conditions and, accordingly by uniqueness, for which assumption III is a sufficient condition, are equal, so that we have

$$-\mathbf{S}\dot{\mathbf{u}}(\mathbf{S}\mathbf{x}) = \dot{\mathbf{u}}(\mathbf{x}), \tag{59}$$

or in component form

$$\dot{u}(x,y,-z) = -\dot{u}(x,y,z), \tag{60}$$

$$\dot{v}(x,y,-z) = -\dot{v}(x,y,z), \tag{61}$$

$$\dot{w}(x,y,-z) = \dot{w}(x,y,z). \tag{62}$$

Once (14) and (59) are established, the general form of the incremental displacement field can be derived in various ways. Therefore, although an approach similar to that adopted in the previous case of extension and bending can be used here, in the following the incremental displacement field is determined through a different procedure.

4.2.1. Pure-torsion

First, the particular case of pure-torsion is analyzed. In this case the internal resultant force is null and the internal resultant moment $\dot{\mathbf{M}}^{+} = \dot{M}_z\mathbf{k}$ is a constant torsion moment. From (15) the incremental strain field is independent on the axial coordinate z, so that its derivative with respect to the axial coordinate vanishes

$$\dot{\mathbf{E}}(x,y,0)_{,z} = \dot{\mathbf{E}}(x)_{,z} = \frac{1}{2}\left(\nabla\dot{\mathbf{u}}(x) + \nabla^{\mathrm{T}}\dot{\mathbf{u}}(x)\right)_{,z} = \frac{1}{2}\left(\nabla\dot{\mathbf{u}}_{,z}(\mathbf{x}) + \nabla^{\mathrm{T}}\dot{\mathbf{u}}_{,z}(\mathbf{x})\right) = \mathbf{0}. \tag{63}$$

Since the strain tensor $(\nabla\dot{\mathbf{u}}_{,z}(\mathbf{x}) + \nabla^{\mathrm{T}}\dot{\mathbf{u}}_{,z}(\mathbf{x}))/2$ is null, we conclude that $\dot{\mathbf{u}}_{,z}(\mathbf{x})$ must be a rigid incremental displacement field

$$\dot{\mathbf{u}}_{,z}(\mathbf{x}) = \dot{\mathbf{u}}_{,z}(\mathbf{0}) + \dot{\boldsymbol{\kappa}} \wedge \mathbf{x}, \tag{64}$$

where $\dot{\boldsymbol{\kappa}}$ denotes a generic infinitesimal incremental rotation vector.

Next, (59) implies that $\dot{\mathbf{u}}_{,z}(\mathbf{x})$ is also a symmetric displacement field

$$\dot{\mathbf{u}}_{,z}(\mathbf{x}) = \mathbf{S}\dot{\mathbf{u}}_{,\mathbf{z}}(\mathbf{S}\mathbf{x}) \tag{65}$$

and then (64) reduces to

$$\dot{\mathbf{u}}_{,z}(\mathbf{x}) = \dot{u}_{,z}(\mathbf{0})\mathbf{i} + \dot{v}_{,z}(\mathbf{0})\mathbf{j} + \dot{\kappa}_z\mathbf{k} \wedge (x\mathbf{i} + y\mathbf{j}). \tag{66}$$

Further, by integrating with respect to z, we obtain

$$\dot{\mathbf{u}}(\mathbf{x}) = \dot{\mathbf{u}}(x,y,0) + \int_0^z \dot{\mathbf{u}}_{,z}(x,y,z)\,\mathrm{d}z = \dot{\mathbf{u}}(x,y,0) + z\dot{u}_{,z}(\mathbf{0})\mathbf{i} + z\dot{v}_{,z}(\mathbf{0})\mathbf{j} + \dot{\kappa}_z z\mathbf{k} \wedge (x\mathbf{i} + y\mathbf{j}), \tag{67}$$

where in view of (59) and (6), it follows that:

$$\dot{\mathbf{u}}(x,y,0^{+}) = -\mathbf{S}\dot{\mathbf{u}}(x,y,0^{-}) = \dot{w}(x,y,0)\mathbf{k}. \tag{68}$$

Hence, we find the general form of the incremental displacement field

$$\dot{\mathbf{u}}(\mathbf{x}) = z\mathbf{k} \wedge [\dot{\kappa}_z(x\mathbf{i} + y\mathbf{j}) + (\dot{v}_{,z}(\mathbf{0})\mathbf{i} - \dot{u}_{,z}(\mathbf{0})\mathbf{j})] + \dot{w}(x,y,0)\mathbf{k}, \tag{69}$$

which depends on the unknown specific angle of twist $\dot{\kappa}_z$ and on the unknown function $\dot{w}(x,y,0)$. The general form (69) shows that a generic cross-section is subject to a rigid rotation along an axis parallel to the cylinder axis, with a twist angle $\dot{\kappa}_z z$ proportional to its axial coordinate z. Further, all sections are subject to a warping displacement $\dot{w}(x,y,0)\mathbf{k}$ out of their planes.

Once the general form (69) is known, the indeterminacy of torsion relaxed equilibrium problem is overcome and its solution can be found in a standard way. First, let us note that the symmetric strains $\dot{\epsilon}_x,\ \dot{\epsilon}_y,\ \dot{\epsilon}_{xy}$ and $\dot{\epsilon}_z$, whose sign is unchanged by reflection $\mathbf{S}$, vanish

$$[\dot{\mathbf{E}}(\mathbf{x})] = \begin{bmatrix} 0 & 0 & \dot{\epsilon}_{xz}(x,y) \\ 0 & 0 & \dot{\epsilon}_{yz}(x,y) \\ \dot{\epsilon}_{xz}(x,y) & \dot{\epsilon}_{yz}(x,y) & 0 \end{bmatrix} \tag{70}$$

and that, in view of I, II and (70), the incremental symmetric stresses also vanish

$$[\dot{\mathbf{T}}(x,y)] = \begin{bmatrix} 0 & 0 & \dot{\tau}_{zx}(x,y) \\ 0 & 0 & \dot{\tau}_{zy}(x,y) \\ \dot{\tau}_{zx}(x,y) & \dot{\tau}_{zy}(x,y) & 0 \end{bmatrix}. \tag{71}$$

In view of (69)–(71) the equilibrium problem (5a)–(5e) reduces to the following plane problem:

$$\dot{\tau}_{zx} = \frac{1}{2}\dot{\kappa}_z\left[\mathbb{C}_{(x,y)_{3131}}(\varphi_{,x} - y) + \mathbb{C}_{(x,y)_{3132}}(\varphi_{,y} + x)\right], \tag{72a}$$

$$\dot{\tau}_{zy} = \frac{1}{2}\dot{\kappa}_z\left[\mathbb{C}_{(x,y)_{3231}}(\varphi_{,x} - y) + \mathbb{C}_{(x,y)_{3232}}(\varphi_{,y} + x)\right], \tag{72b}$$

$$\dot{\tau}_{zx,x} + \dot{\tau}_{zy,y} = 0 \quad \text{in } \Sigma, \tag{72c}$$

$$\dot{\tau}_{zx}n_x + \dot{\tau}_{zy}n_y = 0 \quad \text{in } \partial\Sigma, \tag{72d}$$

$$\dot{M}_z\mathbf{k} = \int_\Sigma (x\mathbf{i} + y\mathbf{j}) \wedge (\dot{\tau}_{zx}\mathbf{i} + \dot{\tau}_{zy}\mathbf{j})\, \mathrm{d}A, \tag{72e}$$

$$\varphi(0,0) = 0, \tag{72f}$$

where $\varphi = \varphi(x,y)$ is the warping function, defined by the relation

$$\dot{\kappa}_z\varphi(x,y) = \dot{w}(x,y,0) + x\dot{u}_{,z}(\mathbf{0}) + y\dot{v}_{,z}(\mathbf{0}). \tag{73}$$

The above plane problem allow us to determine the warping function $\varphi(x,y)$ and the incremental specific angle of twist $\dot{\kappa}_z$ in (69), while the kinematic parameters $\dot{u}_{,z}(\mathbf{0})$ and $\dot{v}_{,z}(\mathbf{0})$ are next determined by imposing the constraints (5e).

4.2.2. Torsion–flexure

We now consider the general antisymmetric problem by including the resultant end shearing forces and so allowing the stress and strain fields to depend upon the axial coordinate. It should be noted that in the general antisymmetric problem the assumption II on the tangent elasticity field has a clear physical interpretation only for linear elastic materials, while for non linear materials generally it is not reasonable. However, in this section we will adopt an incremental formulation, as in previous sections, in order to retain consistency of treatment. From (11) the incremental strain field has in this case the general form:

$$\dot{\mathbf{E}}(x,y,z) = \dot{\mathbf{E}}(x,y,0) + z\dot{\mathbf{E}}'(x,y,0), \tag{74}$$

where $\dot{\mathbf{E}}'(x,y,0)$ denotes the derivative with respect the axial coordinate z of the incremental strain field and, as already stated, it is equal to the incremental strain field induced by the incremental bending moment $\dot{\mathbf{R}}^+ \wedge \mathbf{k}$. Furthermore, (59) gives

$$-\mathbf{S}\dot{\mathbf{E}}(x,y,0)\mathbf{S} = \dot{\mathbf{E}}(x,y,0), \tag{75}$$

which implies

$$\dot{\mathbf{E}}(x,y,0) = \begin{bmatrix} 0 & 0 & \dot{\epsilon}_{xz}(x,y) \\ 0 & 0 & \dot{\epsilon}_{yz}(x,y) \\ \dot{\epsilon}_{xz}(x,y) & \dot{\epsilon}_{xz}(x,y) & 0 \end{bmatrix}, \tag{76}$$

while, (14), (36) and (47) imply that

$$\dot{\mathbf{E}}'(x,y,0) = \begin{bmatrix} \dot{\epsilon}'_x(x,y) & \dot{\epsilon}'_{xy}(x,y) & 0 \\ \dot{\epsilon}'_{xy}(x,y) & \dot{\epsilon}'_y(x,y) & 0 \\ 0 & 0 & \dot{\epsilon}'_0 + \dot{\kappa}'_y x + \dot{\kappa}'_x y \end{bmatrix}, \tag{77}$$

where the derivative of the axial strain with respect to the axial coordinate z depends linearly on the unknown kinematic parameters $\dot{\epsilon}'_0$, $\dot{\kappa}'_y$ and $\dot{\kappa}'_x$. The general form of displacement field can now be determined from (59), (74) and (77). First, from (59), we obtain that the derivative of displacement field with respect to the axial coordinate must be a symmetric displacement field

$$\dot{\mathbf{u}}_{,z}(\mathbf{x}) = \mathbf{S}\dot{\mathbf{u}}_{,z}(\mathbf{S}\mathbf{x}). \tag{78}$$

Further, we have

$$\dot{\mathbf{E}}'(x,y,0) = \frac{1}{2}\left(\nabla\dot{\mathbf{u}}(x) + \nabla^{\mathrm{T}}\dot{\mathbf{u}}(x)\right)_{,z} = \frac{1}{2}\left(\nabla\dot{\mathbf{u}}_{,z}(\mathbf{x}) + \nabla^{\mathrm{T}}\dot{\mathbf{u}}_{,z}(\mathbf{x})\right). \tag{79}$$

Then, as already shown in the case of extension and bending, (77)–(79) allows us to conclude that the derivative with respect to z of the incremental displacement field is

$$\dot{\mathbf{u}}_{,z}(\mathbf{x}) = \left(\dot{u}_{,z}(x,y,0) - \kappa'_y\frac{z^2}{2}\right)\mathbf{i} + \left(\dot{v}_{,z}(x,y,0) - \dot{\kappa}'_x\frac{z^2}{2}\right)\mathbf{j} + (\dot{\epsilon}'_0 + \dot{\kappa}'_y x + \dot{\kappa}'_x y)z\mathbf{k}. \tag{80}$$

By integrating with respect to the axial coordinate, we find

$$\begin{aligned}\dot{\mathbf{u}}(\mathbf{x}) &= \dot{\mathbf{u}}(x,y,0) + \int_0^z \dot{\mathbf{u}}_{,z}(x,y,z)\,\mathrm{d}z \\ &= \dot{\mathbf{u}}(x,y,0) + \left[z\dot{u}_{,z}(x,y,0) - \dot{\kappa}'_y\frac{z^3}{6}\right]\mathbf{i} + \left[z\dot{v}_{,z}(x,y,0) - \dot{\kappa}'_x\frac{z^3}{6}\right]\mathbf{j} + \left[\left(\dot{\varepsilon}'_0 + \dot{\kappa}'_y x + \dot{\kappa}'_x y\right)\frac{z^2}{2}\right]\mathbf{k}.\end{aligned} \tag{81}$$

where in view of (59) and (6), it must be

$$\dot{\mathbf{u}}(x,y,0) = \dot{w}(x,y,0)\mathbf{k}. \tag{82}$$

In conclusion, the general form of the incremental displacement field in the antisymmetric Saint-Venant's problem is

$$\dot{\mathbf{u}}(\mathbf{x}) = \left[z\dot{u}_{,z}(x,y,0) - \dot{\kappa}'_y\frac{z^3}{6}\right]\mathbf{i} + \left[z\dot{v}_{,z}(x,y,0) - \dot{\kappa}'_x\frac{z^3}{6}\right]\mathbf{j} + \left[\left(\dot{\varepsilon}'_0 + \dot{\kappa}'_y x + \dot{\kappa}'_x y\right)\frac{z^2}{2} + \dot{w}(x,y,0)\right]\mathbf{k}, \tag{83}$$

which depends on the unknown kinematic parameters $\dot{\epsilon}'_0$, $\dot{\kappa}'_y$, $\dot{\kappa}'_x$ and on the unknown functions $\dot{u}_{,z}(x,y,0)$, $\dot{v}_{,z}(x,y,0)$ and $\dot{w}(x,y,0)$. Once the above general form is known, the indeterminacy of torsion–flexure relaxed equilibrium problem is overcome and its solution can be found in a standard way. From I, II, (74), (76) and (77) we find the general form of the incremental stress field

$$[\dot{\mathbf{T}}(\mathbf{x})] = \begin{bmatrix} 0 & 0 & \dot{\tau}_{zx}(x,y) \\ 0 & 0 & \dot{\tau}_{zy}(x,y) \\ \dot{\tau}_{zx}(x,y) & \dot{\tau}_{zy}(x,y) & 0 \end{bmatrix} + z\begin{bmatrix} \dot{\sigma}'_x(x,y) & \dot{\tau}'_{xy}(x,y) & 0 \\ \dot{\tau}'_{xy}(x,y) & \dot{\sigma}'_y(x,y) & 0 \\ 0 & 0 & \dot{\sigma}'_z(x,y) \end{bmatrix}. \tag{84}$$

Next, the equilibrium problem (5a)–(5e) reduces to two plane equilibrium problems, to be solved in sequence. With this aim the stress and strain fields are decomposed as follows:

$$\begin{aligned}\dot{\mathbf{E}}(x) &= \dot{\mathbf{E}}(x,y,0) + z\left[\dot{\mathbf{E}}^{\circ\prime}(x,y) + \dot{\varepsilon}'_z(x,y)k\otimes k\right], \\ \dot{\mathbf{T}}(x) &= \dot{\mathbf{T}}(x,y,0) + z\left[\dot{\mathbf{T}}^{\circ\prime}(x,y) + \dot{\sigma}'_z(x,y)k\otimes k\right]\end{aligned} \tag{85}$$

and, in view of (83)–(85), I, II, and by imposing (5a), (5b) and (5c) we are led to the following plane problem, which involves only the symmetric stress and strain components and that is similar to the plane problem (51a)–(51e), already examined in the case of extension and bending

$$\dot{\mathbf{E}}^{\circ\prime} = \dot{u}_{,zx}\mathbf{i}\otimes\mathbf{i} + \dot{v}_{,zy}\mathbf{j}\otimes\mathbf{j} + \frac{(\dot{u}_{,zy} + \dot{v}_{,zx})}{2}(\mathbf{i}\otimes\mathbf{j} + \mathbf{j}\otimes\mathbf{i}), \tag{86a}$$

$$\dot{\mathbf{T}}^{\circ\prime} = \mathbb{C}^{\circ}_{(x,y)}[\dot{\mathbf{E}}^{\circ\prime}] + (\dot{\epsilon}'_0 + \dot{\kappa}'_y x + \dot{\kappa}'_x y)[\mathbb{C}_{(x,y)_{1133}}\mathbf{i}\otimes\mathbf{i} + \mathbb{C}_{(x,y)_{2233}}\mathbf{j}\otimes\mathbf{j} + \mathbb{C}_{(x,y)_{1233}}(\mathbf{i}\otimes\mathbf{j} + \mathbf{j}\otimes\mathbf{i})], \tag{86b}$$

$$\dot{\mathbf{T}}^{\circ\prime}{}_{,x}\mathbf{i} + \dot{\mathbf{T}}^{\circ\prime}{}_{,y}\mathbf{j} = \mathbf{0} \quad \text{in } \Sigma, \tag{86c}$$

$$\dot{\mathbf{T}}^{\circ\prime}(n_x\mathbf{i} + n_y\mathbf{j}) = \mathbf{0} \quad \text{in } \partial\Sigma. \tag{86d}$$

Since no kinematic constraints are imposed, the displacement solution of the above plane problem is determined unless an arbitrary incremental rigid displacement and can be written as:

$$\dot{u}_{,z}(x,y,0)\mathbf{i}+\dot{v}_{,z}(x,y,0)\mathbf{j}=u'(x,y)\mathbf{i}+v'(x,y)\mathbf{j}+\dot{u}_{,z}(\mathbf{0})\mathbf{i}+\dot{v}_{,z}(\mathbf{0})\mathbf{j}+\dot{\kappa}_z\mathbf{k}\wedge(x\mathbf{i}+y\mathbf{j}), \tag{87}$$

where the kinematic parameters $\dot{u}_{,z}(\mathbf{0})$, $\dot{v}_{,z}(\mathbf{0})$ and $\dot{\kappa}_z$ can be arbitrary chosen, while the displacement fields $u'(x,y)$ and $v'(x,y)$ are chosen such as to satisfy the conditions $u'(\mathbf{0})=v'(\mathbf{0})$ and $u'_{,y}(\mathbf{0})=v'_{,x}(\mathbf{0})$.

Due to the particular form of (86b), the displacement fields $u'(x,y)$, $v'(x,y)$ and the strain field $\dot{\mathbf{E}}^{\circ\prime}(x,y)$ depend linearly on the unknown kinematic parameters $\dot{\epsilon}'_0$, $\dot{\kappa}'_x$ and $\dot{\kappa}'_y$

$$u'(x,y)=u'_{\varepsilon}(x,y)\dot{\epsilon}_0+u'_{\kappa_x}(x,y)\dot{\kappa}_x+u'_{\kappa_y}(x,y)\dot{\kappa}_y, \tag{88}$$

$$v'(x,y)=v'_{\varepsilon}(x,y)\dot{\epsilon}_0+v'_{\kappa_x}(x,y)\dot{\kappa}_x+v'_{\kappa_y}(x,y)\dot{\kappa}_y, \tag{89}$$

$$\dot{\mathbf{E}}^{\circ\prime}(x,y)=\mathbf{E}^{\circ\prime}_{\varepsilon}(x,y)\dot{\epsilon}_0+\mathbf{E}^{\circ\prime}_{\kappa_x}(x,y)\dot{\kappa}_x+\mathbf{E}^{\circ\prime}_{\kappa_y}(x,y)\dot{\kappa}_y, \tag{90}$$

where the meaning of the adopted symbols is analogous to that of the previous case of extension and bending. Next, in view of (5d), the unknown kinematic parameters $\dot{\epsilon}'_0$, $\dot{\kappa}'_x$ and $\dot{\kappa}'_y$ can be determined by prescribing in a generic section the following global equilibrium conditions:

$$0=\int_{\Sigma}\dot{\sigma}'_z(x,y)\,\mathrm{d}A,\quad \left(\dot{T}_x\mathbf{i}+\dot{T}_y\mathbf{j}\right)\wedge\mathbf{k}=\int_{\Sigma}(x\mathbf{i}+y\mathbf{j})\wedge\dot{\sigma}'_z(x,y)\mathbf{k}\,\mathrm{d}A \tag{91}$$

where in view of (90), $\dot{\sigma}'_z(x,y)$ depends linearly on $\dot{\epsilon}'_0$, $\dot{\kappa}'_x$ and $\dot{\kappa}'_y$

$$\dot{\sigma}'_z(x,y)=\mathbf{k}\cdot\mathbb{C}_{(x,y)}\left[\mathbf{E}^{\circ\prime}_{\varepsilon}(x,y)\dot{\epsilon}_0+\mathbf{E}^{\circ\prime}_{\kappa_x}(x,y)\dot{\kappa}_x+\mathbf{E}^{\circ\prime}_{\kappa_y}(x,y)\dot{\kappa}_y\right]\mathbf{k}+\mathbb{C}_{(x,y)_{3333}}(\dot{\epsilon}'_0+\dot{\kappa}'_y x+\dot{\kappa}'_x y). \tag{92}$$

Once the first plane equilibrium problem is solved, we are led to a second plane equilibrium problem, which involves the antisymmetric strain and stress components

$$\dot{\tau}_{zx}=\frac{1}{2}\left[\mathbb{C}_{(x,y)_{3131}}[\dot{\kappa}_z(\varphi_{,x}-y)+\dot{u}']+\mathbb{C}_{(x,y)_{3132}}\left[\dot{\kappa}_z\left(\varphi_{,y}+x\right)+\dot{v}'\right]\right], \tag{93a}$$

$$\dot{\tau}_{zy}=\frac{1}{2}\left[\mathbb{C}_{(x,y)_{3231}}[\dot{\kappa}_z(\varphi_{,x}-y)+\dot{u}']+\mathbb{C}_{(x,y)_{3232}}\left[\dot{\kappa}_z\left(\varphi_{,y}+x\right)+\dot{v}'\right]\right], \tag{93b}$$

$$\dot{\tau}_{zx,x}+\dot{\tau}_{zy,y}=-\dot{\sigma}'_z\quad\text{in }\Sigma, \tag{93c}$$

$$\dot{\tau}_{zx}n_x+\dot{\tau}_{zy}n_y=0\quad\text{in }\partial\Sigma, \tag{93d}$$

$$\dot{M}_z\mathbf{k}=\int_{\Sigma}(x\mathbf{i}+y\mathbf{j})\wedge\left(\dot{\tau}_{zx}\mathbf{i}+\dot{\tau}_{zy}\mathbf{j}\right)\mathrm{d}A, \tag{93e}$$

$$\varphi(0,0)=0, \tag{93f}$$

where the warping function $\varphi=\varphi(x,y)$ is defined by the relation (73).

The above problem allows us to determine the warping function φ and the incremental specific angle of twist $\dot{\kappa}_z$. Finally, the parameters $\dot{u}_{,z}(\mathbf{0})$ and $\dot{v}_{,z}(\mathbf{0})$ can be determined by imposing the constraints (5e).

Also in this case we investigate the conditions under which the Clebsh characterization holds. If we assume $\dot{\mathbf{T}}^{\circ\prime}=\mathbf{0}$, we clearly satisfy the equilibrium conditions (86c) and (86d) and we determine the plane incremental strain

$$\dot{\mathbf{E}}^{\circ\prime}_{*}(x,y)=-(\epsilon'^0_z+\kappa'_y x+\kappa'_x y)\mathbb{C}^{\circ^{-1}}_{(x,y)}[\mathbb{C}_{(x,y)_{1133}}\mathbf{i}\otimes\mathbf{i}+\mathbb{C}_{(x,y)_{2233}}\mathbf{j}\otimes\mathbf{j}+\mathbb{C}_{(x,y)_{1233}}(\mathbf{i}\otimes\mathbf{j}+\mathbf{j}\otimes\mathbf{i})], \tag{94}$$

which must satisfy the compatibility condition

$$\dot{\epsilon}^{\circ\prime}_{*x,yy}(x,y)+\dot{\epsilon}^{\circ\prime}_{*y,xx}(x,y)=2\dot{\epsilon}^{\circ\prime}_{*xy,xy}(x,y), \tag{95}$$

identical to the condition (58), already determined in the case of bending and extension.

5. Conclusion

In this work a rational scheme for the solution of a relaxed Saint-Venant's problem is developed. The scheme avoids any semi-inverse procedure and it is valid for a particular heterogeneous anisotropic elastic material. Symmetry is consistently employed to decompose the relaxed equilibrium problem into a symmetric problem and an antisymmetric problem. So that the basic extension and bending solutions are induced by unspecified symmetric surface forces at plane ends, while the basic torsion–flexure and pure-torsion solutions are induced by unspecified antisymmetric surface forces. Then, Saint-Venant's postulate, momentum balance and symmetry are sufficient to determine the general form of the displacement field and also to remove the remaining indeterminacies.

In conclusion, the work is aimed to point out that, although the semi-inverse method is absolutely legitimate it could not be the most straightforward and appropriate procedure to gain a deep insight into this equilibrium problem.

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